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BASICS OF BINARY STAR RESEARCH

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It has been estimated that more than half of the "stars" in our galaxy are multiple systems. Among these, the binary stars are particularly interesting for several reasons. From principles of celestial mechanics, we expect only the binaries to have periodic orbits, and only the binaries should form "close" systems, in which the separation of the components is not very large compared to their sizes. Although such close binaries are relatively uncommon in space, their binary nature can be discovered even at great distances because of their variable line-of-sight velocities and, in many cases, their mutual eclipses, so thousands of such systems have been catalogued. From these catalogs, and from those of the more widely separated binaries, we obtain virtually our only direct data on stellar masses, and most of the accurate determinations of stellar radii and luminosities.

The study of binary stars begins with the problems and principles encountered for single stars and finds many new ones. Some of the main areas include:

1) Decoding of the information in the observational record, such as periodic brightness and radial velocity changes, to determine the present characteristics of particular systems.

2) Theory of the direct physical interactions of binary components, such as the tidal and radiative interactions and the effects of mass exchange.

3) Binary star evolution studies, in which the results of (1) are compared with predictions based on (2) and on single-star evolution theory to discover how mutually interacting stars go through their life cycles.

As a good starting point for thinking about such problems, consider the following simple question. What figures (i.e. shapes) do binary components assume as a result of rotation and tides? This question is a direct part of (2) and, as we shall see, has important consequences with regard to (1) and (3). When the stars are sufficiently well separated so that tidal effects can be ignored it is easy to compute their shapes, but the effects of rotation, tides, and non-circular orbits in combination result in a very difficult problem. Fortunately, even for the closest binaries we find a special case which has a fairly simple solution and which is quite common in nature. This is the case in which the components rotate as our moon does, with the same period as the orbital motion, and this is called synchronous rotation. Synchronous rotation and circular orbits are the rule for very close binaries because tidal drag has the effect of producing just these conditions. The synchronous case is relatively simple because there are no relative motions of any part of the system with respect to any other, so that the forces which arise from the turning of the entire system behave just like static forces. Although the binary system revolves in space as a whole, in effect "it has no moving parts". It next happens that we are allowed to make one further simplifying assumption which, as can be shown, introduces extremely little error. This simplification is that, although the components may be relatively large and considerably distorted, they attract one another nearly as if their entire masses were concentrated into mass-points at their centers.* The mathematics by which this problem is then solved need account only for the gravitational attractions of these two mass points, according to Newton's law of gravitation, and for the force due to the rotation of the binary system about its center of mass.

Before examining the specific results in regard to the figures of binary stars, let us consider a simple example, the rotating earth. What principle governs the particular shape assumed by the earth? As is customary, we adopt mean sea level as defining the figure of the earth. The ocean surface unerringly forms a smooth surface (apart from waves, of course) without hills or valleys, and we wish to consider just how the water arranges itself to do this. In common language, we say that "water seeks its own level". Should any irregularities temporarily be created over the ocean surface, they are removed by flows until no further flows are necessary. Where is the water surface when a steady condition has been reached or, alternatively, what does the water "understand" by

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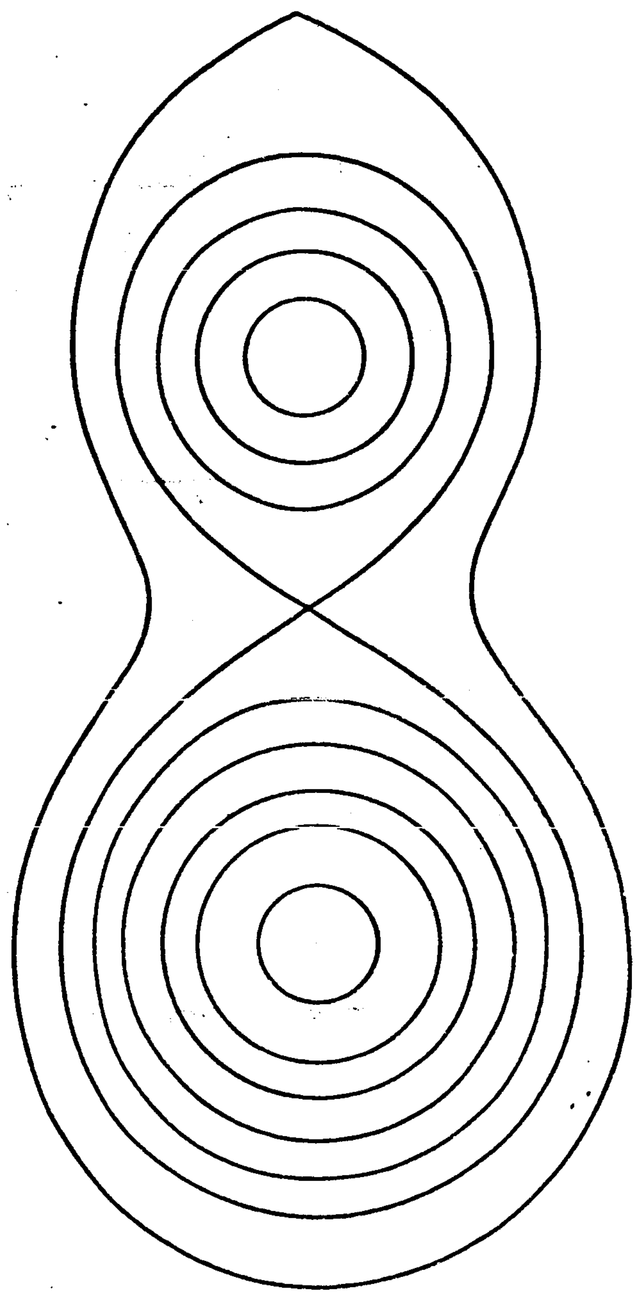
* For a spherical star this rule is exact. That is, the gravitational field outside a spherical star is identical to that which would be produced by an equal mass squeezed down to very small radius. For a star with tidal or rotational distortion, the rule is not far from correct because real stars have most of their mass concentrated into a fairly small high-density core, with the outer, distorted regions having relatively low density.

"its own level"? Well, a level surface in this sense is one on which the potential energy is the same at all places. This will include rotational as well as gravitational potential energy. If the former were zero (no rotation) the earth would be spherical, since surfaces of constant gravitational potential about a mass point or spherical mass are spherical. Rotational forces are, in fact, very small on the earth's surface compared to gravitational forces, so the earth is not far from being spherical.

Suppose now a significant amount of water were added to or taken from the earth's oceans, so that mean sea level became slightly higher or lower. Of course, this new ocean surface would still coincide with a surface of constant potential energy (equipotential surface), but a slightly larger or smaller one. We see, therefore, that there are an infinite number of such surfaces around the earth, a particular one of which happens to mark present-day sea level. The binary star case is similar. The "surface" is a fluid, this time a gas, and we expect the gas to become arranged so as to have constant density along surfaces of constant potential energy. The main complications are that now we have <u>two</u> sources of gravitational attraction and the center of rotation (center of mass) is not at either mass but in between them. However the problem of locating the equipotential surfaces in this case was solved about a century ago by E. Roche. The surfaces have uses other than in specifying the shapes of binary star components, such as in the study of orbits of very small masses in a binary system.* Figure 1 shows some examples of these surfaces. Those which are close to the idealized point masses, M_1 and M_2, are nearly spherical, while those which are successively larger are more and more tidally elongated and rotationally flattened. Therefore stars which are very small compared to their separation will be virtually spherical, while those which are larger will be increasingly egg-shaped (with the inward-facing end smaller than the other, just like an egg) and also rotationally flattened.

What should we expect to find at the place where the family of surfaces around M_1 merges with that around M2? First, consider an intuitively obvious idea. It is certain that there must exist a balance point - including the attractions of M_1 and M_2 and also rotational force - somewhere on the line of centers between the components. At that point, any matter which is somewhow forced to rotate with the system "will not know which way to fall" - it will be in balance. Material slightly closer to M_1 will fall toward M_1 (or if a part of star 1, will remain so), and conversely for material which is slightly closer to M_2. This balance point is called the inner Lagrangian point (L_1 point), after J. L. Lagrange, who

* This is the "restricted 3-body problem" of celestial mechanics, in which the surfaces are usually called zero velocity surfaces.



studied the celestial mechanical aspects of the problem, and has unique significance for our problem. Obviously, if we draw successively larger equipotential surfaces around M_1 we shall eventually draw one which includes the L_1 point, and similarly for M_2. Now the mathematical analysis shows that the largest such surfaces which completely enclose one component or the other are those which include the L_1 point. This is not surprising, for otherwise we could find part of the surface on one component on the side of the balance point toward the other - that is, in a region where it should be gravitationally dominated by the other component. The analysis also gives the detailed shapes of these largest closed equipotential surfaces, and we see that each comes to a point on the inner facing side. These surfaces enclose the Roche lobes for components 1 and 2 and they set the largest dimensions each component can have before starting to spill its material onto the other component. We now see that stars which are successively larger will have greater and greater distortion of figure until they reach the Roche lobe surface when, in effect, a hole opens up at the L_1 point and further size increase is prevented by simple loss of material to the other star. Two interesting points immediately become evident:

1) A binary component which is undergoing a steady expansion (as in normal stellar evolution) will rather accurately assume the dimensions of its Roche lobe since rapid loss of material through the "hole" at the L_1 point prevents achieving a larger size, while the

continuing expansion rules out any smaller size. The situation is very much like the fixed level reached by the water in a tub which has an overflow port.

2) The Roche lobe dimensions can be significantly exceeded only if both components have filled their respective lobes, so that neither can serve as a sink for the material of the other. We have in this case the well-known situation of a contact binary. While most known contact binaries are just barely in contact, in that the stars are only slightly larger than their lobes, a few are known to be substantially larger, and have a thick, connecting neck.

The foregoing points may be summarized in a few statements. The case of synchronous rotation with circular orbits is a common one among close binaries because tidal drag works to establish these conditions. The figures of such stars can be computed quite accurately because synchronous rotation leads to major simplifications in the physical and mathematical treatment, and allows one to consider the "photospheres"* of the components as surfaces of constant potential energy, or "level surfaces". We find that binary components are virtually spherical when they are small compared to their Roche lobes, become progressively more egg-shaped as they approach the size of their Roche lobes, and

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* In present usage a "photosphere" is not necessarily a sphere. The term was first used for the sun, which is essentially spherical.

should actually develop a point on one end when they exactly fill their lobes. Thus the degree of tidal distortion for each component depends on how large it is compared to its own Roche lobe. A Roche lobe may be defined as the volume enclosed by the largest equipotential surface which completely surrounds a given component (but does not enclose the other component). Except for contact binaries, a star cannot be significantly larger than its Roche lobe* because it will, almost immediately, lose any matter which is outside the lobe to the other component. Thus stars undergoing evolutionary expansion will assume the size and shape of their Roche lobes with near-exactness.

Which Roche lobe is the larger? Well, the location of the L_1 point relative to the binary component stars can be computed quite accurately if the mass ratio is known. Table 1 lists a few cases, and we see that L_1 is always closer to the less massive component. A familiar example is provided by the balance point in the earth-moon system, which is rather close to the moon, although in this case the relative position is not fixed because the moon's orbit is eccentric. If the L_1 point is closer to the less massive body, then obviously (cf. Fig. 1) the less massive body must have the smaller Roche lobe.

* It has recently been argued that a significant excess over Roche lobe dimensions is possible in certain brief phases of binary star evolution, but this is not established at present.

Table 1

Location of Balance Point

M2/M1	Relative Distance from Center of Star 1 to L1 Point	Relative Distance from Center of Star 2 to L1 Point
1.0	0.50	0.50
0.8	0.52	0.48
0.5	0.57	0.43
0.2	0.66	0.34
0.1	0.72	0.28

For a specific example, consider the prototype of the eclipsing binaries, Algol, or β Persei. Algol is actually a triple system, but we are interested now only in the eclipsing pair - a nearly spherical B8 main sequence star* whose eclipse provides the main brightness variation, and a tidally distorted sub-giant of spectral type K or G, which causes the main eclipse and is in turn covered by the B8 star in a shallow secondary eclipse. Figure 2 shows a computer-generated "picture" of the Algol system. We see that the components have nearly equal sizes but that one is far more distorted than the other. The reason is easy to state - the spherical B8 star is about 5 times more massive than its cooler companion. We understand this state of affairs because the B8 star, being much the more massive star, has a much larger Roche lobe than the subgiant. It, therefore, is small compared to its lobe whereas the subgiant is large compared to its lobe and in fact, fills it entirely. Of course, even if one knew nothing about the Roche model the situation could easily be rationalized just by saying that the more massive star should produce the larger tides in its companion. However, although this seems reasonable for Algol, imagine a binary in which the massive primary is considerably larger than the Algol primary and the light secondary considerably smaller than the Algol secondary, but with the same 5 to 1 mass ratio. Then we could easily have a case in which the <u>more massive</u> star is the <u>more distorted</u>. Therefore it is neither the mass ratio alone nor the size ratio alone which determines relative distortion, but rather the sizes of the components

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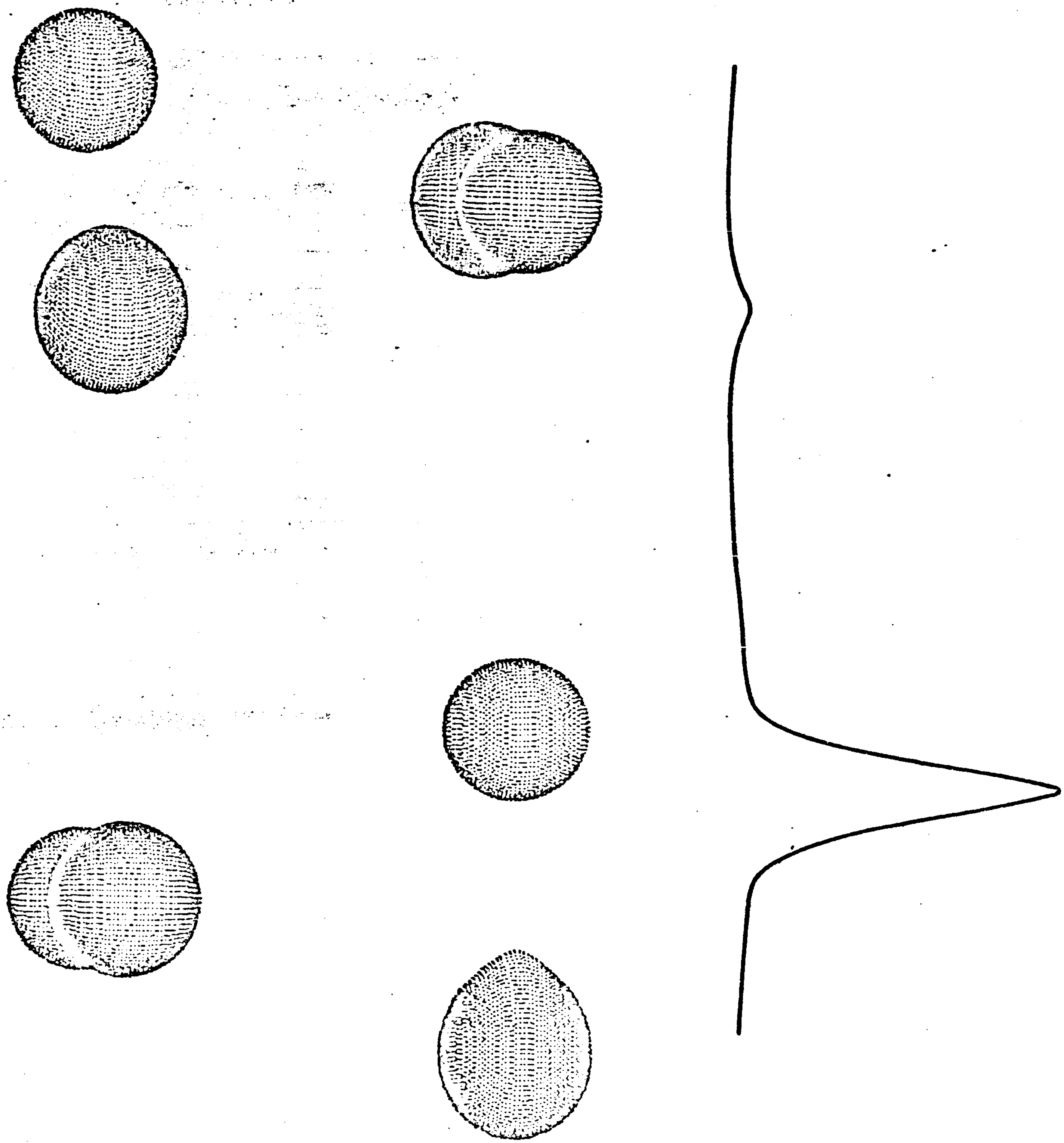
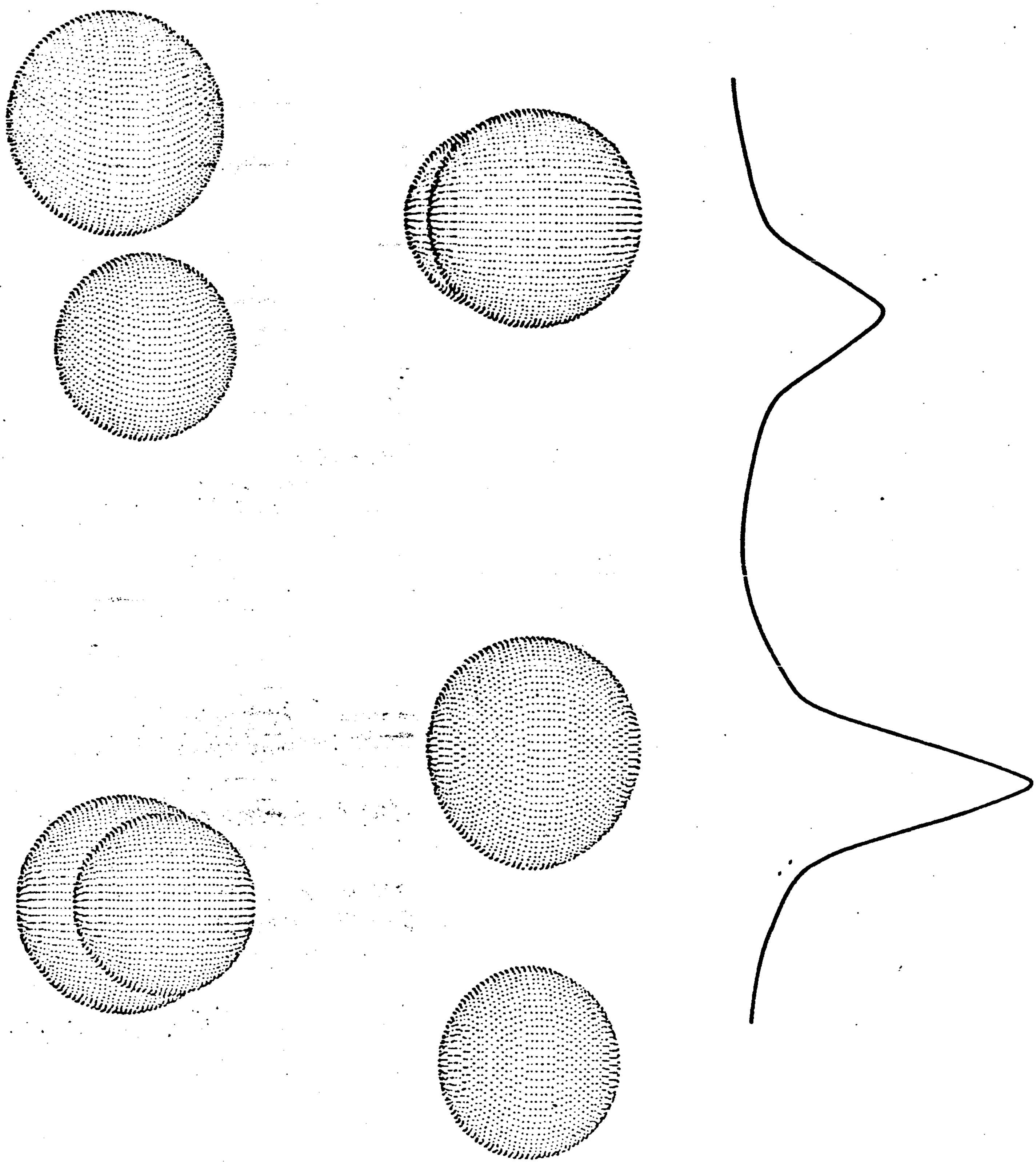
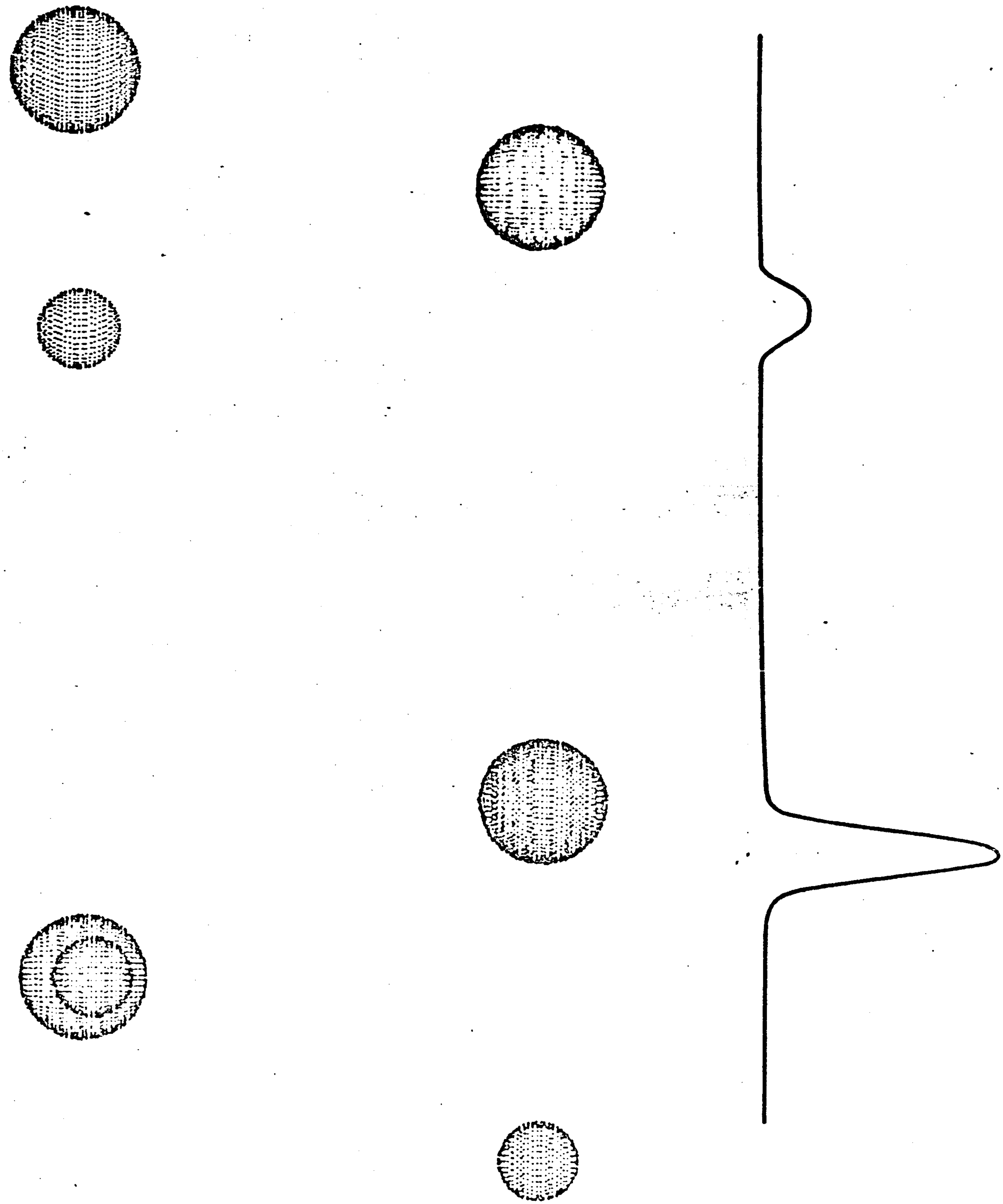
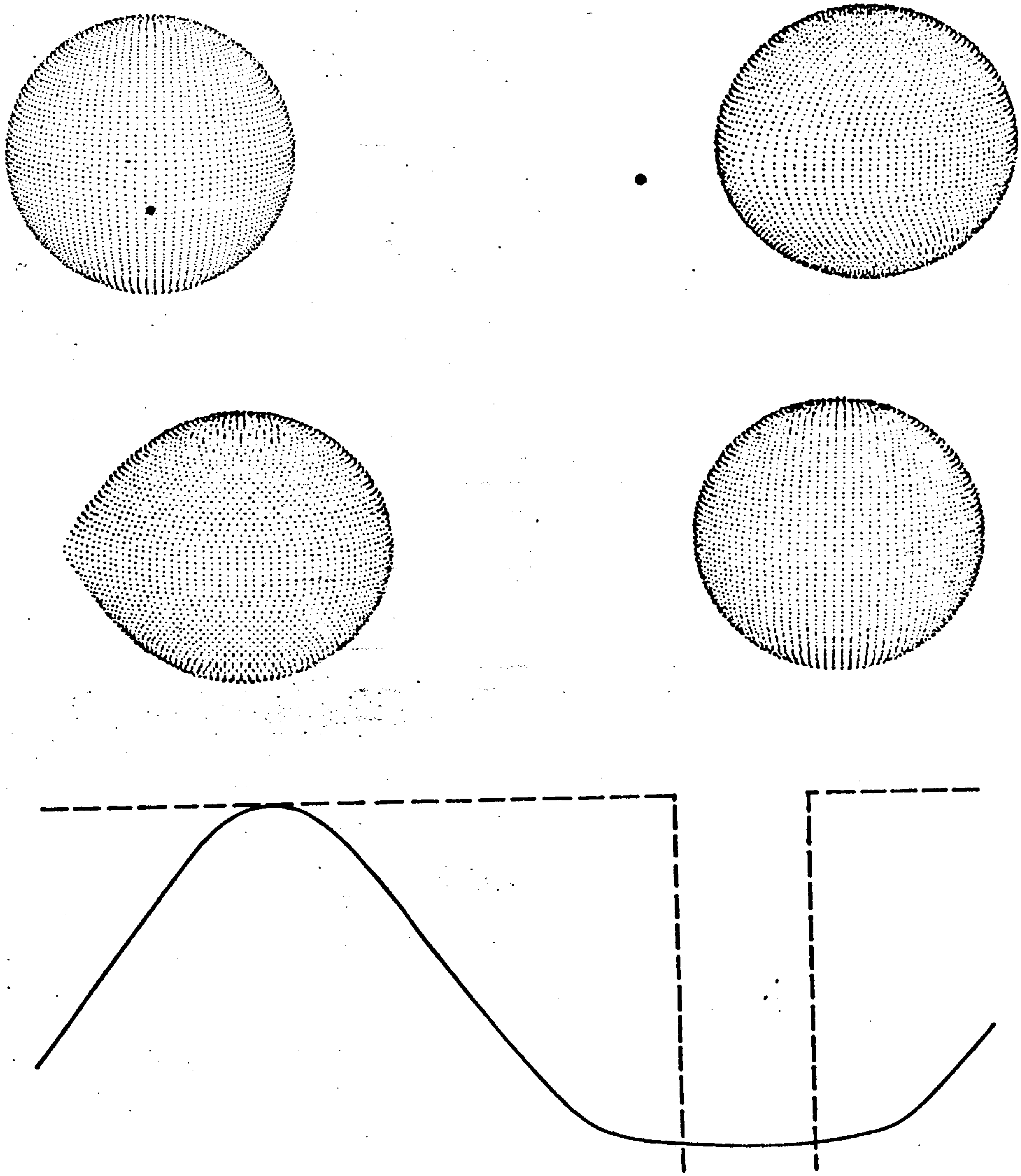
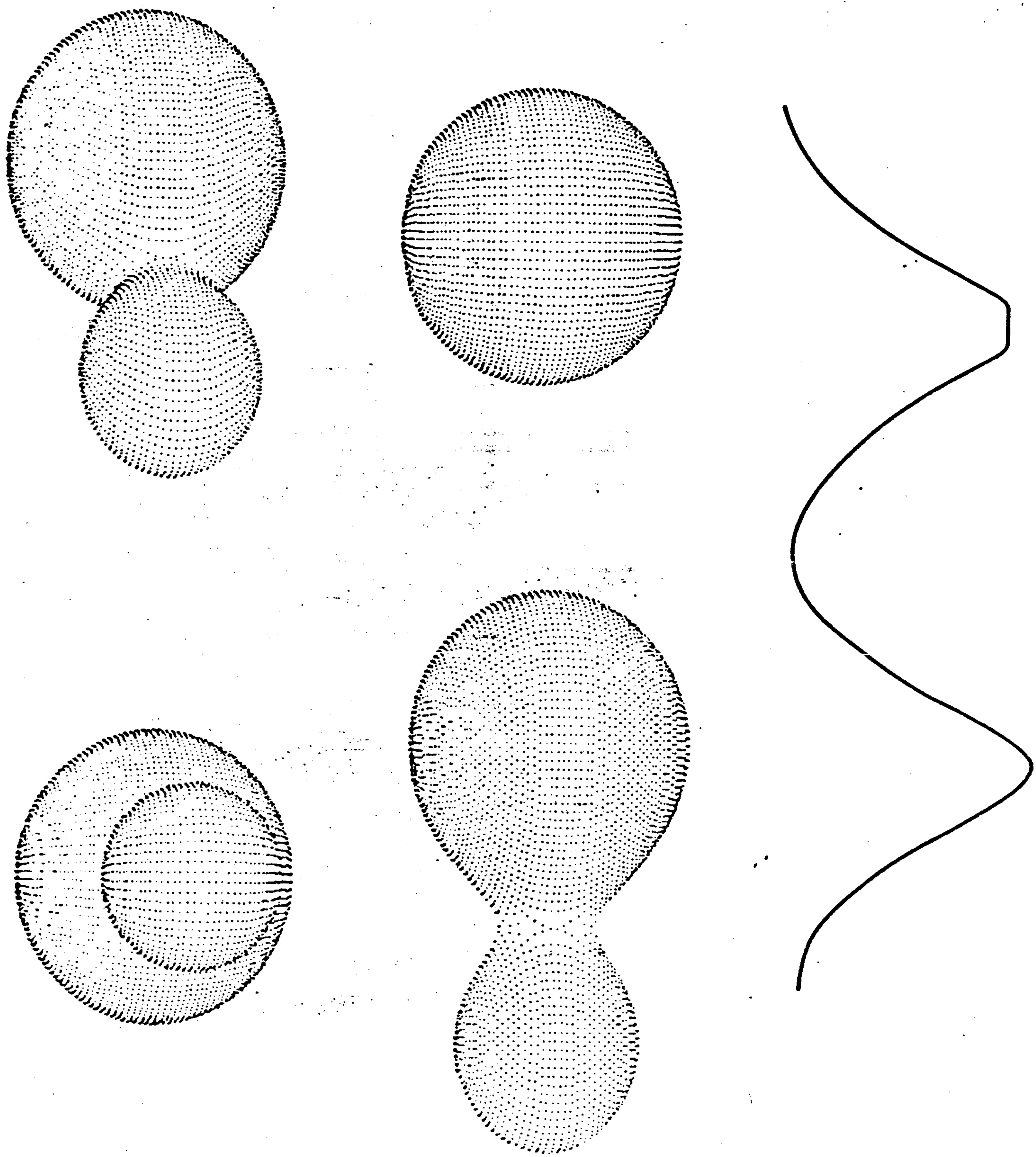
* A star which has not yet begun its evolutionary expansion, or is expanding only very slowly because the hydrogen fuel in the core is not seriously depleted.

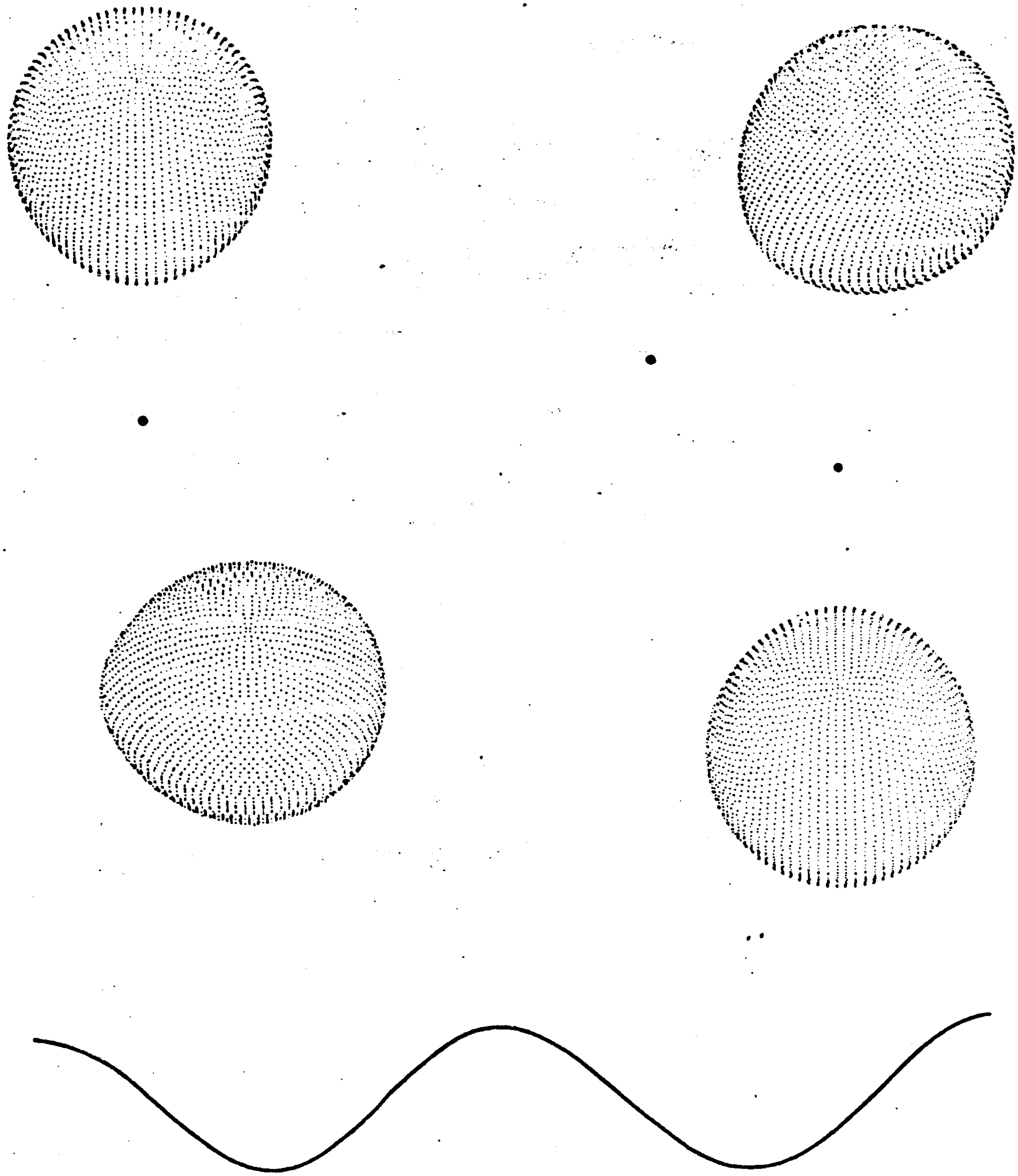












compared to the sizes of their Roche lobes which, in turn, are determined by the mass ratio. Viewed another way, we can say that tides (whether dynamic, as ocean tides, or static, as we are now discussing) are due to differences of gravitational forces. The moon attracts the near side of the earth more strongly than the far side and thus "stretches" the earth (or ocean) along the moon-earth line. We can imagine this tide being increased either by having the moon be more massive, or by having the earth larger in diameter. In the first case, we increase all forces and thus also their differences, while in the second case we increase only the differences, but in either case the tidal effect is increased.

Further examples of binaries in various stages of filling their lobes are given in Figures 3, 4, 5, 6 and 7. Figure 3 shows the binary MR Cygni, which is known as a detached system because each component is "detached" from its Roche lobe. Note that both components show strong tidal distortion, but not such great distortion as the Algol secondary. Algol (Fig. 2), in which one component fills its lobe while the other does not, is an example of a semi-detached system. Figure 4 shows the system of EE Peg, in which both components are small compared to their lobes, and therefore are close to being spherical. Figure 5 shows an estimate of the appearance of HZ Hercules (or Her X-1), the x-ray eclipsing binary discovered in 1971 with the UHURU satellite. The picture shows a star of middle spectral class which is filling its Roche lobe, although it is not certain at this time that the star actually does so - it could be a little smaller. The small dot represents the x-ray component which, at present, is generally believed to be a neutron star.* If the dot were printed to scale, it would be far too small to be seen, since the radius of a neutron star is of the order of 10 kilometers. There is some uncertainty in the masses of the components of HZ Her, but reasonable values would be 1.7 solar masses for the optical star and 0.8 solar masses for the x-ray star. Here we find an example (a rather extreme one) in which it is the more massive component which has the greater tidal distortion. Indeed, the x-ray star could afford to be larger by a considerable factor and still remain essentially spherical, while its more massive companion, being about the size of its Roche lobe, has very large permanent tides.

Figure 6 shows the contact binary RZ Tauri. When both components exceed their lobes, as here, only one surface equipotential can exist so there must be one smooth surface to act as a boundary for the two components, as shown. Obviously, if the contrary were the case (i.e. if the "surface" of one component were at a higher potential level than that of the other) flows would occur between components until such differences were eliminated. Figure 7 shows the x-ray binary Cygnus X-1,

* A neutron star is an object of stellar mass in which the support against self-gravitation is provided by nuclear degenerate pressure. Such stars may be formed in the pre-explosion collapse of supernovae. Neutron stars have been discussed in several semi-popular articles (eg. M. A. Ruderman, Scientific American, February 1972, p. 24).

also known as HDE 226868. In this system there appears to be a good chance that the small x-ray component is a black hole - a star whose final collapse is still proceeding, but has been brought to a virtual halt for outside observers by the gravitational slowing of time. The oscillations of a radiating atom on such an object would be similarly halted, insofar as we could observe, so that the only practical means for detection of black holes would be through their gravitation. At present, the main and perhaps only hope for finding black holes seems to lie in their possible occurrance in binary star systems. Here the gravitational field would be detectable through its effect on the motion of the other binary component, and perhaps through its role in producing x-rays. Mass transfer would be important not only in the evolution of the system but also in generating the x-rays. Transferred gas would become heated to perhaps 50 million degrees in falling near to and into the black hole, and gas at such a temperature radiates primarily x-radiation. The same process works for neutron star and possibly white dwarf binary components.

Each of the Figures (2-7) includes a theoretical light curve showing one cycle of the periodic variation of brightness with time for each system. The effects of tidal distortion are evident for MR Cygni, RZ Tauri, and Cyg X-1 in that these systems show brightness variation even between the eclipses, being brightest when the elongated stars are seen broadside and faintest when the narrow ends are viewed.

That is, the general curvature in the light curves between eclipses is due to tidal distortion. In fact, Cyg X-1 has no eclipses because of the low inclination of about 30°, and its entire variation is due to the ellipticity effect. In addition to this simple variation in visible surface area, which is known as the geometric ellipticity effect, there is a "photometric" ellipticity effect which affects the light curve in a manner very similar to that of geometric ellipticity and which therefore enhances the apparent ellipticity effect. Photometric ellipticity results from the phenomenon of gravity darkening, which causes stellar surfaces to be brightest at the rotational poles and dimmest at the equator. More quantitatively, the local emission of radiant energy per unit area is proportional* to the local gravity, which is also greatest at the poles and least at the equator. Transferring this result to binary stars, we expect again that the poles will be brightest, the equator relatively dim, and the "ends of the egg" dimmest of all.

Five of the binaries exhibit some reflection effect, but it is most easily noticed for HZ Her, Algol, and MR Cyg. It appears mainly as a brightening near the time of secondary eclipse. The secondary eclipse, therefore, appears "at the top of a little hill" on the light curve. This is because the inner-facing side of the cooler component is heated by the strong radiation from the hotter component

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* This relation applies directly only for stars whose internal energy is transported by radiation. For convective cases, it applies in somewhat modified form.

and glows more brightly than the outward-facing side. Near the time of secondary eclipse we are looking directly at this heated side, so the binary system appears especially bright. Of course, the cooler star also heats the hotter one somewhat, but this is a smaller effect unless the components have equal temperatures. For RZ Tau (Figure 6) the two reflection effects virtually cancel one another because the temperatures are nearly equal.

The brightness variation of HZ Her is particularly interesting because it is due almost entirely to the reflection effect. Here the source of heating is the x-radiation from the x-ray star. We see no eclipse in the computed optical light curve because the x-ray star covers only an insignificant fraction of the disk of the normal star, while the eclipse of the x-ray star by the normal star is detectable only in the x-ray observations. Notice the rectangular profile of the x-ray eclipse in the schematic x-ray light curve, which is also shown.* Here there is no evidence of the normal star except for its blocking of the x-radiation. The absence of any transition (partial eclipse) regions shows that the x-ray star is very much smaller than the normal star and that the edge of the normal star is sharply defined.

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* The actual observed x-ray variation is far more complicated than the diagram indicates, and has stimulated much work on physical processes associated with Her X-1. The observed optical light curve differs in important ways from the illustrated computed curve, and several ideas have been advanced regarding the cause of these departures from a simple reflection effect model. However, there is no doubt that the main variation of about 1.5 magnitudes is due to the reflection effect.

Among four of the other binaries we can find examples of quite different types of eclipses. Those of Algol and MR Cygni are partial, in each case showing a rapid decline to minimum followed by an equally rapid recovery. The secondary (shallower) eclipse of EE Peg is total like the x-ray eclipse of HZ Her, but in this case the two components have comparable dimensions so that partial phases precede and follow the (flat) total section. Another total eclipse is shown at the secondary minimum of RZ Tau. The primary eclipse of EE Peg is annular. That is, for an interval near mid-eclipse, the disk of the smaller star is contained (projected) entirely within that of the larger star, just as is the moon during an annular eclipse of the sun, leaving a ring of unobscured surface of the larger star in view. As one may imagine, the eclipse bottom would be flat in this case if the large, eclipsed, star had a uniformly bright surface, but in real stars the phenomenon of limb darkening (darkening toward the edge of the visible disk) provides a rounded bottom, such as that of EE Peg. This effect makes the eclipse noticably different from the (pointed) partial eclipses of Algol and MR Cyg and, of course, from the total secondary eclipses of EE Peg and RZ Tau. Limb darkening occurs because the light emitted in the observer's direction from the limb (edge) of a star comes, for the most part, from relatively high layers in the star's semi-transparent photosphere. Since these high layers are cooler than the deeper layers which are seen when viewing the center of the disk, they radiate less strongly so that the limb appears dark relative to the disk-center.

We have seen how a study of the gravitational interaction of binary star components can lead to an understanding of their figures (shapes) and of the circumstances under which we expect transfer of material from one component to the other. We have seen the consequences of tidal distortion on the observational properties of such binaries. What are the consequences of mass transfer? This is a complex issue, involving many problems which remain to be explored, but if we limit the discussion to a few basic principles, some idea can be given of the important progress in understanding binary star evolution which has been made, mostly within the last decade.

One of the best established rules governing the evolution* of single stars is that all phases of evolution proceed faster for stars of greater mass. In particular, we expect a relatively long quiescent existence for all stars during which the radius and luminosity change only very slowly, followed by a relatively brief interval of rapid expansion, during which the star becomes a red giant, just before the effective "death" of the star. A massive star begins its expansion much sooner than a low-mass star. This makes it quite puzzling to consider the case of Algol, for example, in which the primary component of perhaps 5 solar masses appears to be in the early stages of its evolution while the secondary, of about 1 solar mass, is already undergoing its evolutionary expansion. That is, the secondary now fills its

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* By the term "stellar evolution" we understand the formation, "life cycle", and eventual fate of stars of various masses and chemical compositions.

Roche lobe and we know from spectroscopic observations that it is spilling matter onto the primary through the L_1 point. The solution to this paradox is now understood, due to the work of many astronomers, among whom the names of J. Crawford, D. Morton, R. Kippenhahn, B. Paczynski, and M. Plavec are particularly noteworthy. We now know that in systems such as Algol, the mass transfer has been on such a scale as to reverse the mass ratio. The originally massive primary has lost most of its mass to the originally low-mass secondary so that it was, indeed, the more massive star which began its evolutionary expansion first but, at present, that star is no longer the more massive one because of the exchange of material between components. However several questions arise immediately. Why is the mass transfer on such a large scale, with most of the system mass being involved, and so rapid, with the mass ratio being reversed in only 10,000 to 100,000 years? Why, if most of the mass of the original primary is lost to the secondary is not all of it so lost, thus converting the binary to a single star? In other words, it may seem curious at first sight that the standard mass exchange process is so spectacular as to dump, say, 80% of the mass of one star onto the other, yet stops short of dumping 100 percent.

To see why this is so, consider what happens to the Roche lobe of the original primary component at the beginning of mass exchange. We suppose that the primary has been expanding and has just become as large as its Roche lobe, so that it spills a small amount of mass through the balance (L_1) point onto the other component.

In general we can expect this event to alter not only the mass ratio but also also the period and separation of the two stars, for if the mass ratio changed without corresponding changes in period and separation, the system's total angular momentum would change (which is not permitted, of course, unless some matter leaves the entire system). Now it turns out that the separation must decrease when the flow is from the massive to the low mass star, and this requires a decrease in the size of the primary Roche lobe simply because all dimensions which are related to the components' separation have shrunk. That is, the orbital dimensions, including those of the Roche lobe, are just on a smaller scale than before. Furthermore, we have already noted earlier that the relative size of the Roche lobe depends (only) on the mass ratio, with the star of larger mass having the larger lobe. This is therefore a second reason why the primary Roche lobe will become smaller. These two effects, taken together, cause the primary Roche lobe to shrink significantly with the transfer of a fairly small mass. However this, in turn, leads inevitably to further mass loss because the star, having lost some mass, again finds itself slightly overspilling its lobe. We therefore have what is usually called a positive feedback process in that a small initial transfer of mass leads to conditions which encourage further transfer, and so on until the flow becomes quite large. Such a binary system is said to be in the rapid phase of mass transfer.

We now ask what stops this runaway process before all mass is transferred to the secondary. Recall that the orbital separation shrinks when the transfer is from the more to the less massive star. Naturally, at some stage in the procedure the masses become equal, and after that the components must separate in order to conserve angular momentum because flow will then be from the less to the more massive star. Eventually the primary will find itself in a situation in which it no longer overspills its Roche lobe because the lobe no longer shrinks as matter is transferred.* However, by the time this happens, most of the mass of the primary will have been transferred to the secondary, the mass ratio will have been reversed, and we shall have a system perhaps like that of Algol.

The binary has now reached the end of the rapid phase of mass transfer. Further mass exchange is discouraged because it now tends to make the original primary star smaller than its lobe. However its evolutionary expansion, which started the process, will not have stopped, and will now continue to produce a relatively leisurely mass flow from the original primary to the secondary.† This is the slow phase of mass transfer, which we see today in Algol and many other semi-detached binaries. Subsequent developments in the system will depend to a considerable extent on the particular masses and even on the original chemical composition and internal evolutionary state of the components, but in one way or another they must account for most of the

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* In quantitative work it is necessary to account also for the changing equilibrium radius of the mass-losing star. Only the most essential features of the process are described here.

† Notice that observers would now call the original primary the secondary, and vice versa, because of the mass ratio reversal.

very strange and unusual binaries we see, including those with white dwarf, neutron star on black hole components.

Captions for Figures

1. Equatorial cross-section of the Roche surfaces of constant potential energy. As explained in the text, the shapes of binary stars are defined by such surfaces.

2. Computer generated pictures of the semi-detached system of Algol (the demon star) at phases 0.0 (upper left), 0.125 (upper right), 0.25 (lower left) and 0.50 (lower right). Corresponding pictures in Figures 2-7 have the same phases. In a complete orbit phase runs from 0.0 to 1.0. Algol has experienced a reversal of the mass ratio through evolutionary mass transfer.

3. The detached system MR Cygni. These are hot, blue main sequence stars and must have formed fairly recently.

4. The well-detached system EE Pegasi. These main sequence stars are so far inside their Roche lobes as to be virtually spherical.

5. The x-ray binary Hercules X-1 or HZ Herculis. The large tide is raised by the small orbiting dot, which is thought to be a neutron star, and is the source of x-radiation. If drawn to scale, the dot would be invisible.

6. The contact binary RZ Tauri, which is a typical example of the WUMa class of binaries. The components exchange both energy and material through the connecting neck.

7. The x-ray binary Cygnus X-1, or HDE 226868. There is now a great deal of discussion over whether or not the orbiting dot is a gravitationally collapsed object, or black hole. The orbit is inclined by only about 30° to the plane of the sky, so essentially we are "looking down" on the system. As in Figure 5, the dot would be invisible if drawn to scale. The vertical scale on the light curve has been stretched by a factor of ten relative to the scales on Figures 2–6, because the ellipsoidal variation is very small in this case.